Parametricity

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More people should know more about Parametricity!

- **Philosophy:** The world is more uniform than set-theorists think!
  - cf continuity, homotopy theory, category theory, symmetry
- **Categorically:** The correct approach to contravariance
  - Much better than (strong)-dinaturality
- **Logically:** A sophisticated principle of invariance
  - Excitingly applicable over natural and social sciences
- **Programming:** A theory of refinement
  - Rippling changes to a component though a system

So, how do I use Parametricity to make a fortune?

- **Step 1:** Take a type theory
  - Traditionally System F, but MLTT more so recently.
- **Step 2:** Give a relational interpretation of type theory
  - This exposes structural invariants within type theory
- **Step 3:** Use invariants/uniformities to prove properties
  - Theorems for free, (Di)-Naturality, Initial algebras etc.

Overview of this Course

- **Lecture 1:** Basic Parametricity
  - A concrete model using sets and relations
- **Lecture 2:** Fibrational Parametricity
  - An abstract model based upon fibrations
- **Lecture 3:** Cubical Parametricity
  - From proof-irrelevance, to proof-relevance and on!
- **Lecture 4:** MLTT-Parametricity
  - Parametricity and Dependent Types

Lecture 1: Parametricity via Sets and Relations

- **Thesis:** The world is more uniform than set-theorists think
  - It contains structural constraints (continuity, symmetry ...)
  - In logic and type theory, there is parametricity

- **Polymorphism:** A type constructor $\forall a : \mathsf{Type}.\ T a$.
  - Size $\Rightarrow$ work with an intuitionistic meta-theory
  - We can't look at all types so there must be some uniformity.
  - Eg, how many functions $\forall a. a \to a$
  - Contrast ad-hoc polymorphism/parametric polymorphism

- **Free Theorems:** Parametricity shows that *any* function

$$\mathsf{rev} : \forall a. \mathsf{List}\, a \to \mathsf{List}\, a$$

satisfies the algebraic equation

$$\mathsf{rev}(\mathsf{map}\, f\, xs) = \mathsf{map}\, f\, (\mathsf{rev}\, xs)$$

- **Refinement:** Assume a system $T[X]$ containing a component X.
  - Assume related implementations X_1 and X_2 of X.
  - Are the systems $T[X_1]$ and $T[X_2]$ related?

- **Data Types:** Parametricity ensures System F has products, sums, initial algebras (cf Church encodings), second order existentials and final coalgebras

$$\begin{aligned} A \times B &= \forall X.(A \to B \to X) \to X \\ A + B &= \forall X.(A \to X) \to (B \to X) \to X \\ \mu F &= \forall X.(FX \to X) \to X \\ \exists X.T &= \forall X.(\forall Y.TY \to X) \to X \\ \nu F &= \exists X.X \times (X \to TX) \end{aligned}$$

- **Type Isomorphisms:** Parametricity can be used to isomorphisms such as

$$\forall X.A[X, C \times X] \cong \forall X.A[C \to X, X]$$

Motivation 3: Category Theory

- **Naturality:** All elements α of $\forall X.FX \to GX$ are natural

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ {\scriptstyle \alpha X}\downarrow & & \downarrow{\scriptstyle \alpha Y} \\ GX & \xrightarrow[Gf]{} & GY \end{array}$$

- **Mixed Variance?** What about $ev :: \forall X.\forall Y.(X \to Y) \times X \to Y$
  - Dinaturals and strong dinaturals don't behave well

- **Key Idea:** Parametricity intuitively offers

$$\begin{array}{ccc} FX & \xrightarrow{FR} & FY \\ {\scriptstyle \alpha X}\downarrow & & \downarrow{\scriptstyle \alpha Y} \\ GX & \xrightarrow[GR]{} & GY \end{array}$$

1.1 Syntax of System F

- **Key Idea:** Formalise types via judgements $\Gamma \vdash T : \mathsf{Type}$
  - Variables: $X_1, \ldots X_n \vdash X_i : \mathsf{Type}$
  - Functions: If $\Gamma \vdash U, V : \mathsf{Type}$, then $\Gamma \vdash U \to V : \mathsf{Type}$
  - Forall Types: If $\Gamma, X \vdash T : \mathsf{Type}$, then $\Gamma \vdash \forall X.T : \mathsf{Type}$
  - Judgements for defining terms: $\Gamma, \Delta \vdash t : T$ where we ensure $\Gamma \vdash T : \mathsf{Type}$ and $(x_i : T_i) \in \Delta \Rightarrow \Gamma \vdash T_i : \mathsf{Type}$

- **John Reynolds:** Gave not one, but two semantics called logical relations of the following form. Let Set be a universe of sets.

$$\begin{array}{lcl} [\![T]\!]_0 & \in & \mathsf{Set}^{|\Gamma|} \to \mathsf{Set} \\ [\![T]\!]_1 & \in & \forall \theta_1, \theta_2 \in \mathsf{Set}^{|\Gamma|}. \\ & & \qquad \mathsf{Rel}^{|\Gamma|}(\theta_1, \theta_2) \to \mathsf{Rel}([\![T]\!]_0\theta_1, [\![T]\!]_0\theta_2) \end{array}$$

- **Variables:** Pretty Obvious

$$\begin{aligned} [\![X_1,\ldots,X_n \vdash X_i]\!]_0\theta &= \theta_i \\ [\![X_1,\ldots,X_n \vdash X_i]\!]_1 r &= r_i \end{aligned}$$

- **Arrow Types:** If $\Gamma \vdash U \to V$: Type

$$\begin{aligned} [\![\Gamma \vdash U \to V]\!]_0\theta &= [\![\Gamma \vdash U]\!]_0\theta \to [\![\Gamma \vdash V]\!]_0\theta \\ (f,g) \in [\![\Gamma \vdash U \to V]\!]_1 r \quad &\text{iff} \quad (a,b) \in [\![\Gamma \vdash U]\!]_1 r \Rightarrow \\ & \qquad (fa, gb) \in [\![\Gamma \vdash V]\!]_1 r \end{aligned}$$

- **Key Idea:** Reynolds relational semantics allows us to say
  - related functions map related inputs to related outputs

- **Forall Types I:** $[\![\Gamma \vdash \forall X.T]\!]_0\theta$ is the set

  $\{f : (S : \mathsf{Set}) \to [\![T]\!]_0(\theta, S) | R \in \mathsf{Rel}(A, B) \Rightarrow (fA, fB) \in [\![T]\!]_1(\mathsf{Eq}\theta, R)\}$

  - Parametrically polymorphic functions are ad-hoc functions with a uniformity
  - They map related types (inputs) to related values (outputs)

- **Forall Types II:** $(f, g) \in [\![\Gamma \vdash \forall X.T]\!]_1 r$ iff

$$R : \mathsf{Rel}(A, B) \Rightarrow (fA, gB) \in [\![\Gamma \vdash T]\!]_1(r, R)$$

  - two parametrically polymorphic functions are related iff
  - they map related inputs to related outputs.

Finally, Properties of the Logical Relation

- **Identity Extension Lemma:** A lemma about types

$$[\![\Gamma \vdash T]\!]_1(\mathsf{Eq}\theta) = \mathsf{Eq}([\![\Gamma \vdash T]\!]_0\theta)$$

Equality relations mapped to equality relations

- **Fundamental Theorem:** First give a standard semantics to terms. If $\Gamma, \Delta \vdash t : T$, then

$$[\![\Gamma, \Delta \vdash t : T]\!]_0 : (\theta : \mathsf{Set}^{|\Gamma|}) \to [\![\Gamma \vdash \Delta]\!]_0\theta \to [\![\Gamma \vdash T]\!]_0\theta$$

and then prove that

  - if $\theta_1, \theta_2 \in \mathsf{Set}^{|\Gamma|}$ and $r \in \mathsf{Rel}^{|\Gamma|}(\theta_1, \theta_2)$, and if
  - $a_1 \in [\![\Gamma \vdash \Delta]\!]_0\theta_1$ and $a_2 \in [\![\Gamma \vdash \Delta]\!]_0\theta_2$ then
  - $(a_1, a_2) \in [\![\Gamma \vdash \Delta]\!]_1 r \Rightarrow ([\![t]\!]_0\theta_1 a_1, [\![t]\!]_0\theta_2 a_2) \in [\![\Gamma \vdash T]\!]_1 r$

Terms map related inputs to related outputs

- **Theorem:** If F is positive and f is a morphism, then

$$\mathsf{gr}([\![\mathsf{F}]\!]_0 \mathsf{f}) = [\![\mathsf{F}]\!]_1 (\mathsf{gr}\ \ \mathsf{f})$$

- **Theorem:** $\forall X.X \to X = 1$
  - Proof:

- **Theorem:** All elements of $\forall X.FX \to GX$ are natural
  - Proof:

- **Key Idea:** Use IEL and interesting graph relations!

Lecture 2: Fibrational Parametricity

- **Question:** Who likes Type Theory?
  - Well, it has some uses as we have seen
  - But as formulae grow, they get hard to manipulate
  - And, more advanced systems and notions of relation?

- **Goal:** Categorify to understand and generalise
  - A respectful categorical abstraction of what the above constructions actually amount to
  - Lets abstract them so they can be generalised to other calculi
  - And lets have some diagrams!

- **Defn:** A categorical abstraction of a domain of computation and a logic over it. For us, Set and Rel
  - A category B, called the base and a category E, called the total category. A functor $p : E \to B$ mapping each logical formula to the object it is a property of.
  - Define E_B to be those objects of E mapped by p to B
  - Every $f : B \to B'$ defines a functor $f^* : E_{B'} \to E_B$
- **Added Structure:** Truth and opreindexing
  - Truth: Each fibre has a terminal object $\top_B$
  - Opreindexing: Each $f : B \to B'$ is such that f^* has a left adjoint Σ_f

A Fibrational Semantics of Types

- **Fibrations:** Define some categories
  - Set is the category of small sets and functions. Rel has as objects binary relations and as morphisms, pairs of functions between the carriers of the relations preserving relatedness. $p : \mathsf{Rel} \to \mathsf{Set} \times \mathsf{Set}$ maps $R : \mathsf{Rel}(X, Y)$ to (X, Y).

- **Semantics of Types:** If $\Gamma \vdash T : \mathsf{Type}$, and $n = |\Gamma|$, then

$$\begin{CD} |\mathsf{Rel}|^n @>{[\![T]\!]_1}>> \mathsf{Rel} \\ @V{|p|^n}VV @VV{p}V \\ |\mathsf{Set}|^n \times |\mathsf{Set}|^n @>>{[\![T]\!]_0 \times [\![T]\!]_0}> \mathsf{Set} \times \mathsf{Set} \end{CD}$$

- **Key Idea**: No action of type semantics on morphisms!!! And can generalise to all fibrations!

- **Definition:** Equality defines a functor $\mathsf{Eq} : \mathsf{Set} \to \mathsf{Rel}$

- **Identity Extension Lemma:** Simply ...

$$\begin{array}{ccc} |\mathsf{Rel}|^n & \xrightarrow{[\![T]\!]_1} & \mathsf{Rel} \\ \uparrow{\scriptstyle |\mathsf{Eq}|^n} & & \uparrow{\scriptstyle \mathsf{Eq}} \\ |\mathsf{Set}|^n & \xrightarrow[{[\![T]\!]_0}]{} & \mathsf{Set} \end{array}$$

- **Why Fibrations:** Equality can be defined in any bifibration with fibred terminal objects

$$\mathsf{Eq} X = \Sigma_{\delta : X \to X \times X} \top X$$

- **Arrow Types:** The logical relation $R \to R'$ is simply the exponential in Rel.
  - Logical relations are not ad-hoc but fundamental structure

- **$\forall$-types:** Strengthen notion of cone to remove non-parametric elements
  - A T-cone with vertex X is a collection of maps $X \to [\![T]\!]_0 Y$ for every Y. Terminal such are the ad-hoc polymorphic functions.
  - An T-eqcone with vertex X is a collection of maps $\alpha_Y : X \to [\![T]\!]_0 Y$ for every Y, and for every $R : \mathsf{Rel}(X, Y)$, a map $\alpha_R : \mathsf{EqX} \to [\![T]\!]_1 R$ over (α_X, α_Y)
  - The parametric elements are those in the terminal T-eqcone

- **Recall:** The standard interpretation of a term $\Gamma, \Delta \vdash t : T$ is a function

$$[\![\Gamma, \Delta \vdash t : T]\!]_0 : (\theta : \mathsf{Set}^{|\Gamma|}) \to [\![\Gamma \vdash \Delta]\!]_0 \theta \to [\![\Gamma \vdash T]\!]_0 \theta$$

or, categorically:

$$[\![\Gamma, \Delta \vdash t : T]\!]_0 : \mathsf{Nat}\ \ [\![\Gamma \vdash \Delta]\!]_0\ \ [\![\Gamma \vdash T]\!]_0$$

- **Question:** But what about the fundamental theorem ... its just a natural transformation

$$[\![\Gamma, \Delta \vdash t : T]\!]_1 : \mathsf{Nat}\ \ [\![\Gamma \vdash \Delta]\!]_1\ \ [\![\Gamma \vdash T]\!]_1$$

over $[\![\Gamma, \Delta \vdash t : T]\!]_0 \times [\![\Gamma, \Delta \vdash t : T]\!]_0$

- **Key Idea:** Types and terms are not interpreted as functors and natural transformations, but fibred functors and fibred natural transformations

- **Recall:** Reynolds solved the contravariance problem by ditching the action on morphisms. Surely cheating!
  - But every function $f : A \to B$ defines a graph $\mathsf{gr} f : \mathsf{Rel} A B$
  - Reynolds key insight: replace the action of $[\![T]\!]_0$ on f with an action of $[\![T]\!]_1$ on $\mathsf{gr} f$

- **Fibrationally:** Define $\mathsf{gr} : \mathsf{Set}^{\to} \to \mathsf{Rel}$ by
  - $\mathsf{gr} f = (f, id_B)^* \mathsf{Eq} B$, or
  - $\mathsf{gr} f = \Sigma_{(id_A, f)} \mathsf{Eq} A$.
  - Equivalent with BC.

- **Graph Lemma:** We need both directions of the graph lemma
  - Reindexing gives $[\![F]\!]_1(\mathsf{gr} f) \to \mathsf{gr}([\![F]\!]_0 f)$
  - Opreindexing gives $\mathsf{gr}([\![F]\!]_0 f) \to [\![F]\!]_1(\mathsf{gr} f)$

- **Theorem:** $\mathsf{gr} : \mathsf{Set}^{\to} \to \mathsf{Rel}$ is full and faithful when Eq is.
  - So not only do we trade morphisms in the base for objects in the total category, but ...
  - ... we trade commuting squares in the base of morphisms in the total category

- **Related work:** Hermida kicked off fibrational parametricity
  - Birkedal, Mogelberg, Simpson, Dunphy/Reddy
  - Us: bifibrations for the graph lemma, universal characterisation of parametric elements.

- **Future:** Clean enough to travel many directions including
  - Higher Dimensional Parametricity and intensional MLTT
  - Parametricity for Symmetry
  - Parametricity in the Natural Sciences